On a surprising relation between rectangular and square free convolutions

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ON A SURPRISING RELATION BETWEEN RECTANGULAR AND SQUARE FREE CONVOLUTIONS

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Abstract. Debbah and Ryan have recently [DR07] proved a result about the limit empirical singular distribution of the sum of two rectangular random matrices whose dimensions tend to infinity. In this paper, we reformulate it in terms of the rectangular free convolution introduced in [BG07b] and then we give a new, shorter, proof of this result under weaker hypothesis: we do not suppose the probability measure in question in this result to be compactly supported anymore. At last, we discuss the inclusion of this result in the family of relations between rectangular and square random matrices.

Introduction

Free convolutions are operations on probability measures on the real line which allow to compute the spectral or singular empirical measures of large random matrices which are expressed as sums or products of independent random matrices, the spectral measures of which are known. More specifically, the operations $\boxplus, \boxtimes$, called respectively *free additive and multiplicative convolutions* are defined in the following way [VDN91]. Let, for each n, M_n, N_n be n by n independent random hermitian matrices, one of them having a distribution which is invariant under the action of the unitary group by conjugation, which empirical spectral measures[1] converge, as n tends to infinity, to non random probability measures denoted respectively by τ_1, τ_2. Then $\tau_1 \boxplus \tau_2$ is the limit of the empirical spectral law of $M_n + N_n$ and, in the case where the matrices are positive, $\tau_1 \boxtimes \tau_2$ is the limit of the empirical spectral law of $M_n N_n$. In the same way, for any $\lambda \in [0, 1]$, the *rectangular free convolution* $\boxplus_\lambda$ is defined, in [BG07b], in the following way. Let $M_{n,p}, N_{n,p}$ be n by p independent random matrices, one of them having a distribution which is invariant by multiplication by any unitary matrix on any side, which symmetrized[2] empirical singular measures[3] tend, as n, p tend to infinity in such a way that n/p tends to λ, to non random probability measures ν_1, ν_2. Then the symmetrized empirical singular law of $M_{n,p} + N_{n,p}$ tends to $\nu_1 \boxplus_\lambda \nu_2$.

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[1]The *empirical spectral measure* of a matrix is the uniform law on its eigenvalues with multiplicity.

[2]The *symmetrization* of a probability measure μ on $[0, +\infty)$ is the law of εX, for ε, X independent random variables with respective laws $\frac{\delta_1 + \delta_{-1}}{2}, \mu$. Dealing with laws on $[0, +\infty)$ or with their symmetrizations is equivalent, but for historical reasons, the rectangular free convolutions have been defined with symmetric laws. In all this paper, we shall often pass from symmetric probability measures to measures on $[0, +\infty)$ and vice-versa. Thus in order to avoid confusion, we shall mainly use the letter μ for measures on $[0, \infty)$ and ν for symmetric ones.

[3]The *empirical singular measure* of a matrix M with size n by p ($n \leq p$) is the empirical spectral measure of $|M| := \sqrt{MM^*}$.

These operations can be explicitly computed using either a combinatorial or an analytic machinery (see [VDN91] and [NS06] for $\boxplus, \boxtimes$ and [BG07b] for $\boxplus_\lambda$). In the cases $\lambda = 0$ or $\lambda = 1$, i.e. where the rectangular random matrices we consider are either "almost flat" or "almost square", the rectangular free convolution with ratio λ can be expressed with the additive free convolution: $\boxplus_1 = \boxplus$ and for all symmetric laws ν_1, ν_2, $\nu_1 \boxplus_0 \nu_2$ is the symmetric law which push-forward by the map $t \mapsto t^2$ is the free convolution of the push forwards of ν_1 and ν_2 by the same map. However, though one can find many analogies between the definitions of $\boxplus$ and $\boxplus_\lambda$ and still more analogies have been proved [BG07a], no general relation between $\boxplus_\lambda$ and $\boxplus$ had been proved until a paper of Debbah and Ryan [DR07] (which submitted version, more focused on applications than on this result, is [DR08]). It is to notice that this result is not due to researchers from the communities of Operator Algebras or Probability Theory, but to researchers from Information Theory, working on communication networks. In [DR07], Debbah and Ryan proved a result about random matrices which can be interpreted as an expression, for certain probability measures ν_1, ν_2, of their rectangular convolution $\nu_1 \boxplus_\lambda \nu_2$ in terms of $\boxplus$ and of another convolution, called the *free multiplicative deconvolution* and denoted by $\boxbslash$. In this note, we present this result with a new approach and we give a new and shorter proof, where the hypothesis are more general. This generalization of the hypothesis answers a question asked by Debbah and Ryan in the last section of their paper [DR07]. The question of a more general relation between square and rectangular free convolutions is considered in a last "perspectives" section.

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1. The result of Debbah and Ryan

Let us define the operation $\boxbslash$ on certain pairs of probability measures on $[0, +\infty)$ in the following way. For μ, μ_2 probability measures on $[0, +\infty)$, if there is a probability measure on $[0, +\infty)$ such that $\mu = \mu_1 \boxtimes \mu_2$, then μ_1 is called the *free multiplicative deconvolution* of μ by μ_2 and is denoted by $\mu_1 = \mu \boxbslash \mu_2$. Let us define, for $\lambda \in (0, 1]$, μ_λ be the law of λX for X random variable distributed according to the Marchenko-Pastur law with parameter $1/\lambda$, i.e. the law with support $[(1 - \sqrt{\lambda})^2, (1 + \sqrt{\lambda})^2]$ and density

$$x \mapsto \frac{\sqrt{4\lambda - (x - 1 - \lambda)^2}}{2\pi\lambda x}.$$

Theorem 1 of [DR07] states the following result. $\lambda \in (0, 1]$ is fixed and (p_n) is a sequence of positive integers such that n/p_n tends to λ as n tends to infinity. δ_1 denotes the Dirac mass at 1.

Theorem 1 (Debbah and Ryan)**.** *Let, for each n, A_n, G_n be independent n by p_n random matrices such that the empirical spectral law of $A_n A_n^*$ converges almost surely weakly, as n tends to infinity, to a compactly supported probability measure μ_A and such that the entries of G_n are independent $N(0, \frac{1}{p_n})$ random variables. Then the empirical spectral law of $(A_n + G_n)(A_n + G_n)^*$ converges almost surely to a compactly supported probability measure ρ which, in the case where $\mu_A \boxbslash \mu_\lambda$ exists, satisfies the relation*

$$\rho = [(\mu_A \boxbslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda. \tag{1}$$

Remark 2. Note that in the case where $\mu_A \boxbslash \mu_\lambda$ doesn't exist, the relation (1) stays true in the formal sense. More specifically, for μ_A probability measure such that $\mu_A \boxbslash \mu_\lambda$ exists, the moments of $\rho = [(\mu_A \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda$ have a polynomial expression in the moments of μ_A (this can easily be seen by the theory of free cumulants [NS06]). It happens that this relation between the moments of the limit spectral law ρ of $(A_n + G_n)(A_n + G_n)^*$ and the ones of μ_A stays true even when $\mu_A \boxslash \mu_\lambda$ doesn't exist. It follows from the original proof of Theorem 1 and it will also follow from our proof (see Remark 3).

Note that by the very definition of the rectangular free convolution $\boxplus_\lambda$ with ratio λ recalled in the introduction and since the limit empirical spectral law of GG^* is μ_λ (it is a well known fact, see, e.g. Theorem 4.1.9 of [HP00]), this result can be stated as follows: for all compactly supported probability measure μ on $[0, +\infty)$ such that $\mu \boxslash \mu_\lambda$ exists,

$$(\sqrt{\mu} \boxplus_\lambda \sqrt{\mu_\lambda})^2 = [(\mu \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda, \tag{2}$$

where for any probability measure ρ on $[0, +\infty)$, $\sqrt{\rho}$ denotes the symmetrization of the push-forward by the square root functions of ρ and for any symmetric probability measure ν on the real line, ν^2 denotes the push-forward of ν by the function $t \mapsto t^2$. This formula allows to express the operator $\boxplus_\lambda \sqrt{\mu_\lambda}$ on the set of symmetric compactly supported probability measures on the real line in terms of $\boxplus$ and $\boxtimes$: for all symmetric probability measure on the real line ν,

$$\nu \boxplus_\lambda \sqrt{\mu_\lambda} = \sqrt{[(\nu^2 \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda}. \tag{3}$$

2. A proof of the generalized theorem of Debbah and Ryan

$\lambda \in (0, 1]$ is still fixed. In this section, we shall give a new shorter proof of the theorem of Debbah and Ryan, under weaker hypothesis: we shall prove (3) without supposing the support of ν to be compactly supported. The proof is based on the machinery of the rectangular free convolution and of the rectangular R-transform.

2.1. **Some analytic transforms.**

Let us first recall a few facts about the analytic approach to $\boxtimes$ and $\boxplus_\lambda$. Let us define, for ρ probability measure on $[0, \infty)$,

$$M_\rho(z) := \int_{t \in \mathbb{R}} \frac{zt}{1 - zt} \mathrm{d}\rho(t), \quad S_\rho(s) = \frac{1 + z}{z} M_\rho^{\langle -1 \rangle}(z),$$

where, as it shall be in the rest of the text, the exponent ${}^{\langle -1 \rangle}$ stands for the inversion of analytic functions on $\mathbb{C}\backslash[0, +\infty)$ with respect to the composition operation $\circ$, in a neighborhood of zero. By [VDN91], for all pair μ_1, μ_2 of probability measures on $[0, +\infty)$, $\mu_1 \boxtimes \mu_2$ is characterized by the fact that $S_{\mu_1 \boxtimes \mu_2} = S_{\mu_1} S_{\mu_2}$.

In the same way, the rectangular free convolution with ratio λ can be computed with an analytic transform of probability measures. Let ν be a symmetric probability measure on the real line. Let us define $H_\nu(z) = z(\lambda M_{\nu^2}(z) + 1)(M_{\nu^2}(z) + 1)$. Then the *rectangular R-transform with ratio* λ of ν is defined to be

$$C_\nu(z) = U\left(\frac{z}{H_\nu^{\langle -1 \rangle}(z)} - 1 \right),$$

where $U(z) = \frac{-\lambda - 1 + \left[(\lambda+1)^2 + 4\lambda z\right]^{1/2}}{2\lambda}$. By theorem 3.12 of [BG07b], for all pair ν_1, ν_2 of symmetric probability measures, $\nu_1 \boxplus_\lambda \nu_2$ is characterized by the fact that $C_{\nu_1 \boxplus_\lambda \nu_2} = C_{\nu_1} + C_{\nu_2}$.

2.2. **Some preliminary computations.** Note that by [NS06], the S- and R-transforms of a probability measure μ on $[0,+\infty)$ are linked by the relation $S_\mu(s) = \frac{1}{z}R_\mu^{\langle -1\rangle}(z)$, thus since the free cumulants of the Marchenko-Pastur law with parameter $1/\lambda$ are all equal to $1/\lambda$ (see [NS06]), we have $S_{\mu_\lambda}(z) = \frac{1}{1+\lambda z}$. Moreover, since by [NS06] again, $S_\mu(s) = \frac{1+z}{z}M_\mu^{\langle -1\rangle}(z)$, for any law σ on $[0,+\infty)$,

$$M^{\langle -1\rangle}_{\sigma\boxtimes\mu_\lambda} = \frac{z}{z+1}\frac{S_\sigma}{1+\lambda z} = \frac{M_\sigma^{\langle -1\rangle}}{1+\lambda z} \quad \text{and} \quad M^{\langle -1\rangle}_{\sigma\boxslash\mu_\lambda} = (1+\lambda z)M_\sigma^{\langle -1\rangle}. \tag{4}$$

At last, since $\boxplus\delta_1 = *\delta_1$, which implies that $M_{\sigma\boxplus\delta_1}(z) = [(z+1)M_\sigma(z)+z]\circ\frac{z}{z+1}$, for any symmetric law ν, we have

$$M^{\langle -1\rangle}_{((\nu^2\ltimes\mu_\lambda)\boxplus\delta_1)\boxtimes\mu_\lambda} = \frac{1}{1+\lambda z}\times\frac{z}{1+z}\circ\left[(z+1)\left((1+\lambda z)M_{\nu^2}^{\langle -1\rangle}\right)^{\langle -1\rangle}+z\right]^{\langle -1\rangle}. \tag{5}$$

2.3. **Proof of the result.** So let us consider a symmetric probability measure ν such that $\nu^2\boxslash\mu_\lambda$ exists and let us prove (3). As proved in the proof of Theorem 3.8 of [BG07b], for any symmetric probability measure τ, H_τ characterizes τ, thus it suffices to prove that $H_{\nu\boxplus_\lambda\sqrt{\mu_\lambda}} = H_m$ for $m = \sqrt{[(\nu^2\boxslash\mu_\lambda)\boxplus\delta_1]\boxtimes\mu_\lambda}$. By Theorem 4.3 and the paragraph preceding in [BG07a], $C_{\sqrt{\mu_\lambda}}(z) = z$. Thus Lemma 4.1 of [BBA07] applies here, and it states that in a neighborhood of zero in $\mathbb{C}\backslash[0,+\infty)$,

$$H_{\nu\boxplus_\lambda\sqrt{\mu_\lambda}} = H_\nu\circ\left(\frac{H_\nu}{T(H_\nu+M_{\nu^2})}\right)^{\langle -1\rangle},$$

where $T(z) = (\lambda z+1)(z+1)$. So it suffices to prove that in such a neighborhood of zero,

$$H_m = H_\nu\circ\left(\frac{H_\nu}{T(H_\nu+M_{\nu^2})}\right)^{\langle -1\rangle}, \quad \text{i.e.} \quad H_m\circ\frac{H_\nu}{T(H_\nu+M_{\nu^2})} = H_\nu.$$

Using the fact that for any symmetric law τ, $H_\tau(z) = zT(M_{\tau^2}(z))$, it amounts to prove that

$$\frac{H_\nu}{T(H_\nu+M_{\nu^2})}\times T\circ M_{m^2}\circ\frac{H_\nu}{T(H_\nu+M_{\nu^2})} = H_\nu(z),$$

i.e.

$$T\circ M_{m^2}\circ\frac{H_\nu}{T(H_\nu+M_{\nu^2})} = T(H_\nu(z)+M_{\nu^2}(z)),$$

which is implied, simplifying by T and using again $H_\tau(z) = zT(M_{\tau^2}(z))$, by

$$M_{m^2}\circ\frac{zT(M_{\nu^2}(z))}{T[zT(M_{\nu^2}(z))+M_{\nu^2}(z)]} = zT[M_{\nu^2}(z)]+M_{\nu^2}(z).$$

It is implied, composing by $M_{\nu^2}^{\langle -1\rangle}$ on the right and by $M_{m^2}^{\langle -1\rangle}$ on the left, by

$$M_{\nu^2}^{\langle -1\rangle}\times T = (T\times M_{m^2}^{\langle -1\rangle})\circ(M_{\nu^2}^{\langle -1\rangle}(z)T(z)+z).$$

Using the expression of $M_{m^2}^{\langle -1\rangle}$ given by (5), it amounts to prove that

$$M_{\nu^2}^{\langle -1\rangle}(z)T(z) =$$

$$\left((z+1)\times\frac{z}{1+z}\circ\left[(z+1)\left((1+\lambda z)M_{\nu^2}^{\langle -1\rangle}\right)^{\langle -1\rangle}+z\right]^{\langle -1\rangle}\right)\circ(M_{\nu^2}^{\langle -1\rangle}(z)T(z)+z),$$

i.e. that

$$\frac{M_{\nu^2}^{\langle -1\rangle}(z)T(z)}{M_{\nu^2}^{\langle -1\rangle}(z)T(z)+z+1} = \frac{z}{1+z} \circ \left[(z+1)\left((1+\lambda z)M_{\nu^2}^{\langle -1\rangle}\right)^{\langle -1\rangle} + z\right]^{\langle -1\rangle} \circ (M_{\nu^2}^{\langle -1\rangle}(z)T(z)+z).$$

Now, composing by $\left[(z+1)\left((1+\lambda z)M_{\nu^2}^{\langle -1\rangle}\right)^{\langle -1\rangle} + z\right] \circ \frac{z}{1-z}$ on the left, it gives

$$\left[(z+1)\left((1+\lambda z)M_{\nu^2}^{\langle -1\rangle}\right)^{\langle -1\rangle} + z\right] \circ [(1+\lambda z)M_{\nu^2}^{\langle -1\rangle}(z)] = M_{\nu^2}^{\langle -1\rangle}(z)T(z)+z,$$

i.e.

$$[M_{\nu^2}^{\langle -1\rangle}(z)(\lambda z+1)+1]z + [M_{\nu^2}^{\langle -1\rangle}(z)(\lambda z+1)] = M_{\nu^2}^{\langle -1\rangle}(z)(\lambda z+1)(z+1)+z,$$

which is easily verified.

2.4. **Remarks on this result.**

Remark 3. Note that we did not use the fact that $\nu^2 \boxslash \mu_\lambda$ exists to prove that $H_{\nu\boxplus_\lambda\sqrt{\mu_\lambda}} = H_m$. It means that if $\nu^2 \boxslash \mu_\lambda$ doesn't exist, there is no more probability measure μ on $[0,+\infty)$ such that $M_\mu^{\langle -1\rangle} = (1+\lambda z)M_{\nu^2}^{\langle -1\rangle}$ as in (4), but the polynomial expression of the moments of $\nu \boxplus_\lambda \sqrt{\mu_\lambda}$ (i.e. of the limit symmetrized singular law of the matrix $A_n + G_n$ of Theorem 1) in the moments of ν following from $H_{\nu\boxplus_\lambda\sqrt{\mu_\lambda}} = H_m$ for $m = \sqrt{[(\nu^2 \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda}$ stays true (see Remark 2).

Remark 4 (Case $\lambda = 0$)**.** A continuous way to define μ_λ for any $\lambda \in [0,1]$ is to define it to be the probability measure with free cumulants $k_n(\mu_\lambda) = \lambda^{1-n}$ for all $n \geq 1$ (see [NS06]). This definition gives $\mu_0 = \delta_1$. Note that by definition of the rectangular free convolution with null ratio $\boxplus_0$ (which is recalled in the introduction), the relation (3) stays true for $\lambda = 0$.

Remark 5. Note that the original proof of Debbah and Ryan in [DR07] is based on the combinatorics approach to freeness, via the free cumulants of Nica and Speicher [NS06], whereas our proof is based on the analytical machinery for the computation of the rectangular R-transform, namely the rectangular R-transform. It happens sometimes that combinatorial proofs can be translated on the analytical plan by considering the generating functions of the combinatorial objects in question. Notice however that it is not what we did here. Indeed, the rectangular R-transform machinery is actually related to other cumulants than the ones of Nica and Speicher. These are the so-called rectangular cumulants, defined in [BG07b].

2.5. **Remarks about the free deconvolution by μ_λ.** The following corollary is part of the answer given in the present paper to the question asked in the last section of the paper of Debbah and Ryan [DR07]. Let us endow the set of probability measures on the real line with the weak topology [B68].

Corollary 6. *The functional $\nu \mapsto [(\nu^2 \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda$, defined on the set of probability measures ν on $[0,+\infty)$ such that $\nu \boxslash \mu_\lambda$ exists, extends continuously to the whole set of probability measures on $[0,+\infty)$.*

Proof. We just proved, in section 2.3, that the formula

$$\nu \boxplus_\lambda \sqrt{\mu_\lambda} = \sqrt{[(\nu^2 \boxslash \mu_\lambda) \boxplus \delta_1] \boxtimes \mu_\lambda}$$

is true for any probability measure ν on $[0,+\infty)$. Since the operation $\boxplus_\lambda$ is continuous on the set of symmetric probability measures on the real line (Theorem 3.12 of [BG07b]) and the bijective corespondance between symmetric laws on the real line and laws on $[0,+\infty)$, which maps any symmetric law to its push-forward by the map $t \mapsto t^2$, is continuous with continuous inverse, the corollary is obvious. $\square$

The functional $\boxslash \mu_\lambda$, which domain is contained in the set of probability measures on $[0,+\infty)$, plays surprisingly a key role here. It seems natural to try to study its domain. The first step is to notice that this domain is the whole set of probability measures on $[0,+\infty)$ if and only if δ_1 is in this domain, and that in this case, the functional $\boxslash \mu_\lambda$ is simply equal to $\boxtimes(\delta_1 \boxslash \mu_\lambda)$. However, the following proposition states that despite the previous corollary, the domain of the functional $\boxslash \mu_\lambda$ is not the whole set of probability measures on $[0,+\infty)$.

Proposition 7. *The Dirac mass δ_1 at 1 is not in the domain of the functional $\boxslash \mu_\lambda$.*

Proof. Suppose that there is a probability measure τ on $[0,+\infty)$ such that $\delta_1 = \tau \boxtimes \mu_\lambda$. Such a law τ has to satisfy $S_\tau(z) = 1 + \lambda z$. It implies that for z small enough, $M_\tau(z) = \frac{z-1+[(1-z)^2+4\lambda z]^{1/2}}{2\lambda}$. Such a function doesn't admit any analytic continuation to $\mathbb{C}\backslash[0,+\infty)$, thus no such probability measure τ exists. $\square$

3. Relations between square and rectangular matrices/convolutions

The theorem of Debbah and Ryan gives an expression of the empirical singular measure of the sum of two rectangular random matrices in terms of operations related to hermitian square random matrices. Two other results relate empirical singular measures of (non hermitian) square or rectangular random matrices to the operations devoted to hermitian random matrices.

The first one can be resumed by $\boxplus_1 = \boxplus$. Concretely, it states that denoting by $\mathrm{ESM}(X)$ the symmetrization of empirical singular measure of any rectangular matrix X, for any pair M, N of large n by p random matrices, one of them being invariant in law by the left and right actions of the unitary groups, for $n/p \simeq 1$,

$$\mathrm{ESM}(M+N) \simeq \mathrm{ESM}(M) \boxplus \mathrm{ESM}(N). \tag{6}$$

Note that the matrices M, N are not hermitian, which makes (6) pretty surprising (since $\boxplus$ was defined with hermitian random matrices). It means that for $\varepsilon, \varepsilon_1, \varepsilon_2$ independent random variables with law $\frac{\delta_{-1}+\delta_1}{2}$ independent of M and N, we have

$$\mathrm{Spectrum}(\varepsilon|M+N|) \simeq \mathrm{Spectrum}(\varepsilon_1|M| + \varepsilon_2|N|) \tag{7}$$

The second one can be resumed by for any pair ν, τ of symmetric probability measures on the real line, $(\nu \boxplus_0 \tau)^2 = \nu^2 \boxplus \tau^2$ Concretely, it states that for any pair M, N of n by p unitarily invariant random matrices, for $1 << n << p$,

$$\mathrm{Spectrum}[(M+N)(M+N)^*] \simeq \mathrm{Spectrum}(MM^* + NN^*). \tag{8}$$

The advantage of the result of Debbah and Ryan on those ones is that it works for any value of the ratio λ, but its disadvantage is that it only works when one of the laws convoluted is μ_λ, i.e. one of the matrices considered is a Gaussian one. In fact this sharp restriction can be understood by the fact that among rectangular random matrices which are invariant in law under multiplication by unitary matrices, the Gaussian ones are the only ones which can be extended to square matrices which are also invariant in law under multiplication by unitary matrices.

It could be interesting to understand better how relations like (7), (8) or like the one of the Debbah and Ryan's theorem work and can be generalized. Unfortunately, until now, even though nice proofs (see [BG07b] for (7) and (8) or Theorem 4.3.11 of [HP00] and Proposition 3.5 of [HL00] for the $n = p$ case of (7)) relying in free probability have been given for these results relating rectangular convolutions and "square non hermitian convolutions" with the "square hermitian convolution" (i.e. $\boxplus$), no "concrete" explanation has been given, and no generalization (to any λ, to any pair of probability measures) neither. Such a generalization could be the given of a functional f_λ on the set of symmetric probability measures such that for all ν, τ symmetric probability measures, $\nu \boxplus_\lambda \tau$ is the only symmetric probability measure satisfying

$$f_\lambda(\nu \boxplus_\lambda \tau) = f_\lambda(\nu) \boxplus f_\lambda(\tau).$$

Note that in the case $\lambda = 1$, the functional $f_\lambda(\nu) = \nu$ works, and in the case $\lambda = 0$, the functional which maps a measure to its push-forward by the square function works.

Remark 8. Let $(\mathcal{A}, \varphi)$ be a $*$-non commutative probability space and p_1, p_2 be two self-adjoint projectors of $\mathcal{A}$ such that $p_1 + p_2 = 1$ such that $\lambda = \varphi(p_1)/\varphi(p_2)$. As explained in Proposition-Definition 2.1 of [BG07b], $\boxplus_\lambda$ can be defined by the fact that for any pair $a, b \in p_1\mathcal{A}p_2$ free with amalgamation over $\operatorname{Vect}(p_1, p_2)$, the symmetrized distribution of $|a+b|$ in $(p_1\mathcal{A}p_1, \frac{1}{\varphi(p_1)}\varphi_{|p_1\mathcal{A}p_1})$ is the rectangular free convolution with ratio λ of the symmetrized distributions of $|a|$ and $|b|$ in the same space.

Moreover, it is easy to see that for all $a \in p_1\mathcal{A}p_2$, the symmetrized distribution τ of $|a|$ in $(p_1\mathcal{A}p_1, \frac{1}{\varphi(p_1)}\varphi_{|p_1\mathcal{A}p_1})$ is linked to the distribution ν of $a + a^*$ in $(\mathcal{A}, \varphi)$ by the relation $\nu = \frac{2\lambda}{1+\lambda}\tau + \frac{1-\lambda}{1+\lambda}\delta_0$.

When $\lambda = 1$, the equation $\boxplus = \boxplus_\lambda$ can be summurized in the following way: for $a, b \in p_1\mathcal{A}p_2$ free with amalgamation over $\operatorname{Vect}(p_1, p_2)$, the distribution of $(a+b) + (a+b)^*$ in $(\mathcal{A}, \varphi)$ is the free convolution of the distributions of $a + a^*$ and $b + b^*$.

If this had stayed true for other values of λ, it would have meant that for all ν, τ compactly supported symmetric probability measures on the real line, we have

$$f_\lambda(\nu \boxplus_\lambda \tau) = f_\lambda(\nu) \boxplus f_\lambda(\tau), \tag{9}$$

where f_λ is the function which maps a probability measure τ on the real line to $\frac{2\lambda}{1+\lambda}\tau + \frac{1-\lambda}{1+\lambda}\delta_0$. But looking at fourth moment, it appears that (9) isn't true.

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